Cosmology

Christian G. Böhmer*
Department of Mathematics, University College London,
Gower Street, London, WC1E 6BT, UK

March 27, 2009

Abstract

This version of the lecture notes has been typed and prepared by Nyein Chan (nyein.chan@ucl.ac.uk), Department of Physics and Astronomy, University College London, based on the handwritten lecture notes of Christian Böhmer used during the academic year 2007-2008.

The first half of the lectures contains standard material on classical cosmology, that is the Friedmann equations and their solutions in a radiation and matter dominated universe, solutions in the presence of the cosmological constant and the relation of the mathematical quantities to observable quantities. Various distances are discussed in a cosmological context.

The second half addressed modern cosmology, this means cosmological inflation driven by a scalar field, slow-roll parameters and power-law inflation. The final chapter introduces cosmological perturbation theory and describes its use in modern cosmology.

Note that figures are still missing in these notes. Please inform me about typos, flaws or inaccuracies as this is not yet the final version.

*c.boehmer@ucl.ac.uk

Contents

Recommended books

S. Dodelson
Modern Cosmology
Academic Press

A. Liddle and D. Lyth
Cosmological Inflation and Large-Scale Structure
Cambridge University Press

S. Carroll
Spacetime and Geometry
Addison Wesley

S. Carroll
Lecture notes on general relativity
http://arxiv.org/abs/gr-qc/9712019

J. Plebanski and A. Krasinski
An Introduction to General Relativity and Cosmology
Cambridge University Press

1 Cosmological models

1.1 Preliminaries

Recall the field equations of general relativity

$$G_{ab} = 8\pi T_{ab} \tag{1.1}$$

which describe the interplay between matter and geometry.

The key question of classical cosmology is: Which solutions of the Einstein field equations describe the (idealised) universe that we observe?

In order to make qualitative statements about the cosmos, we assume the following three axioms:

Axiom 1. The laws of physics we know are valid everywhere and at all times.

Axiom 2. The fundamental constants, c, G, $\hbar$ etc., are true constants (in particular, it excludes time-varying fundamental constants).

Axiom 3. The universe is connected.

1.2 Introduction

Cosmological observations show that the number of galaxies in any volume of size of about 100MPc is roughly the same. Hence, the universe looks uniform at scales of about 100MPc (1pc = 3.26ly). Therefore, we assume that the universe looks the same everywhere over large enough scales and thus we assume the universe to be homogeneous and isotropic.

Definition 1.1. Homogeneity. A system is homogeneous if it is invariant under translations $X^a \longmapsto X^a + X^c$.

Definition 1.2. Isotropy. A system is isotropic if it is invariant under rotations. It looks the same in all directions.

We are interested in a special set of observers who see an isotropic universe. This defines a special reference frame.

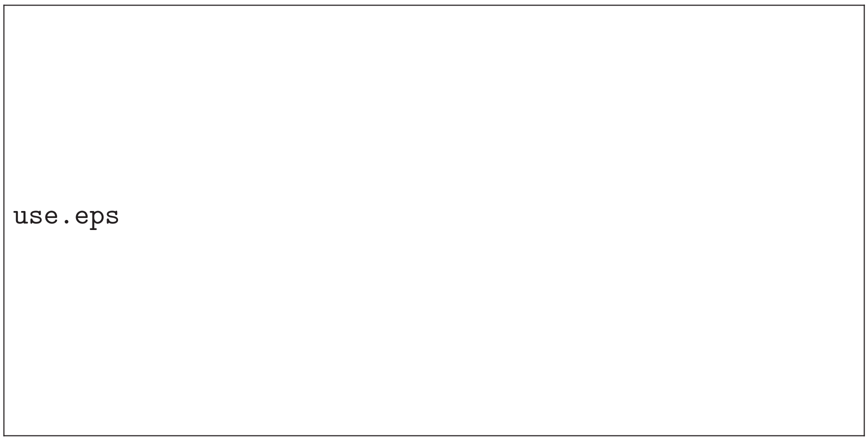
Definition 1.3. Cosmological reference frame. A set of coordinates in which physical quantities are homogeneous and isotropic.

Definition 1.4. Comoving observer. An observer at rest in the cosmological reference frame.

Definition 1.5. Cosmic time. The proper time t measured by a comoving observer, starting with $t = 0$ at the big bang.

1.3 The Friedmann-Lemaître-Robertson-Walker metric

At cosmic time t, the universe defines a three dimensional manifold, a space-like hypersurface.



Homogeneity implies that the universe expands uniformly. The expansion rate is the same everywhere. Homogeneity also implies that the constant time hypersurfaces are spaces of constant curvature. Any two such spaces of the same dimension and the same metric signature have equal constant curvature and are locally isometric.

Spaces of positive constant curvature are spheres, spaces of vanishing constants curvature are ordinary Euclidean flat space and spaces of negative constant curvature are hyperboloids.

These considerations yield the following metric ansatz to describe a homogeneous and isotropic universe

$$ds^2 = -dt^2 + a(t)^2 \frac{dx^2 + dy^2 + dz^2}{(1 + \frac{k}{4}(x^2 + y^2 + z^2))^2} \tag{1.2}$$

$a(t)$ is called the expansion parameter.

Let us denote the spatial part of the line element by $\gamma_{ij} dX^i dX^j$ such that

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij} dX^i dX^j \tag{1.3}$$

where $X^i = (x, y, z)$ and $i = 1, 2, 3$.

The Ricci scalar of the spatial part of the metric is given by

$${}^{(3)}R = \frac{6k}{a^2} \tag{1.4}$$

Therefore the sign of k determines the sign of the spatial Ricci scalar and hence fixes the geometry of the constant time hypersurfaces. However, the complete four-dimensional Ricci scalar reads

$$\begin{aligned} R &= \frac{6k}{a^2} + 6\frac{a'^2}{a^2} + 6\frac{a''}{a} \\ &= {}^{(3)}R + 6\left(\frac{a'^2}{a^2} + \frac{a''}{a}\right) \end{aligned} \tag{1.5}$$

and its sign is not determined by k. Its value at time t depends on the dynamics described by $a(t)$.

1.4 Geometry of constant time hypersurfaces

Let us introduce spherical polar coordinates:

$$x = r\sin\theta\cos\phi, \tag{1.6}$$

$$y = r\sin\theta\sin\phi, \tag{1.7}$$

$$z = r\cos\theta \tag{1.8}$$

which leads to the following spatial metric

$$\gamma_{ij}dX^i dX^j = \frac{dr^2 + r^2 d\Omega^2}{(1+\frac{k}{4}r^2)^2} \tag{1.9}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ is the line element of the unit 2-sphere S^2.

Introducing the new radial coordinate

$$\rho = r\left(1+\frac{k}{4}r^2\right)^{-1}, \tag{1.10}$$

gives

$$\gamma_{ij}dX^i dX^j = \frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega^2. \tag{1.11}$$

For spaces of constant positive curvature $k = 1$ we introduce the third angle of the three sphere

$$\rho = \sin\chi \tag{1.12}$$

which results in

$$\gamma_{ij}dX^i dX^j = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.13}$$

This is the line element of $\mathbb{S}^3$ (the 3-sphere). Every $\chi = chi_0$ =const or $\theta = \theta_0$ =const or $\phi = \phi_0$ =const hypersurface corresponds to a 2-sphere. Cuts through 3-spheres yield 2-spheres, likewise, cuts through 2-spheres yield circles.

To see that this metric indeed describes S^3, consider the flat space $\mathbb{R}^4$ with coordinates (x, y, z, w) and

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{1.14}$$

and choose

$$x = \sin\chi \sin\theta \cos\phi, \tag{1.15}$$
$$y = \sin\chi \sin\theta \sin\phi, \tag{1.16}$$
$$z = \sin\chi \cos\theta, \tag{1.17}$$
$$w = \cos\chi. \tag{1.18}$$

One verifies that

$$x^2 + y^2 + z^2 + w^2 = 1 \tag{1.19}$$

and also

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.20}$$

Let us consider the volume of the compact S^3:

$$\begin{aligned} V &= \int \sqrt{\det \gamma_{ij}} d^3x \\ &= \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^2\chi \sin\theta d\chi d\theta d\phi = 2\pi^2 \end{aligned} \tag{1.21}$$

For flat spaces with $k = 0$ we simply take $\rho = \chi$ and obtain Euclidean three-space in spherical polar coordinates

$$\gamma_{ij} dX^i dX^j = d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.22}$$

For spaces of constant negative curvature $k = -1$, we introduce a 'third angle' of the so-called hyperspace H^3.

$$\rho = \sinh\chi \tag{1.23}$$

which gives

$$\gamma_{ij} dX^i dX^j = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.24}$$

Let us now consider the flat four dimensional Lorentzian manifold:

$$ds^2 = dx^2 + dy^2 + dz^2 - dw^2 \tag{1.25}$$

and take coordinates

$$x = \sinh\chi \sin\theta \cos\phi, \tag{1.26}$$
$$y = \sinh\chi \sin\theta \sin\phi, \tag{1.27}$$
$$z = \sinh\chi \cos\theta, \tag{1.28}$$
$$w = \cosh\chi. \tag{1.29}$$

This yields

$$ds^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.30}$$

the volume of this space is unbounded.

In summary we may write the Friedmann-Lemaître-Robertson-Walker (FLRW) metric in the following form:

$$ds^2 = -dt^2 + a(t)^2 \left[d\chi^2 + \Sigma(\chi)^2 d\Omega^2\right] \tag{1.31}$$

where

$$\Sigma(\chi) = \begin{cases} \sin\chi & \text{if } k = +1 \\ \chi & \text{if } k = 0 \\ \sinh\chi & \text{if } k = -1 \end{cases} \tag{1.32}$$

1.5 Particle horizons

When analysing cosmological models the following question naturally arises: How much of our universe can be observed in principle at a given event p?

In cosmology, the test particles and observers are assumed to be the galaxies. Hence we wish to know which observers (isotropic) could have sent a signal which reaches another observer at or before the event p.

The boundary between world lines that reach p and those that cannot is called the particle horizon at p.

Let us assume a universe of finite age and consider the flat FLRW metric

$$ds^2 - dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{1.33}$$

Make the coordinate transformation

$$\tau = \int \frac{dt}{a(t)}, \qquad d\tau = \frac{dt}{a(t)}. \tag{1.34}$$

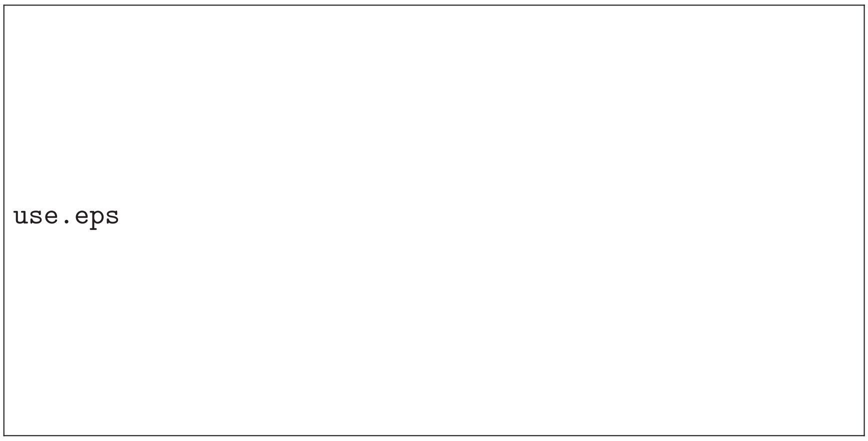
Then the metric becomes

$$ds^2 = a^2(\tau)[-d\tau^2 + dx^2 + dy^2 + dz^2]. \tag{1.35}$$

Hence, this metric is a multiple of the flat Minkowski metric and all coordinates range from $-\infty$ to $+\infty$!

Let us assume the universe began at $t = 0$ and consider an observer at even p. If τ diverges as $t \to 0$, the observer will be able to receive signals from all other observers. The integral diverges if $a(t) \leq \alpha t$ for $\alpha = const$ as $t \to 0$ and there will be no particle horizon since τ will range down to $-\infty$.

If, on the other hand, the integral converges, there exists a particle horizon in the universe because only a portion of τ is covered.



1.6 Matter content and field equations

Matter in the Einstein field equations is described by the stress-energy tensor T_{ab}. On a cosmological scale, each galaxy can be idealised as a test particle or a grain of dust. The pressure induced by the random velocities of the galaxies in negligible. Hence, the present universe can be described by the following stress-energy tensor

$$T_{ab} = \rho u_a u_b \tag{1.36}$$

where u_a is the particle four velocity and $\rho = \rho(t)$ is the mass density of the matter.

Also a thermal distribution of radiation at a temperature of about $3K$ fills the universe. Such radiation is described by an equation of state $P = \frac{\rho}{3}$ and a stress-energy tensor

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \tag{1.37}$$

The cosmological constant Λ or dark energy is characterised by the equation of state

$$w = \frac{P}{\rho} = -1 \tag{1.38}$$

A few commonly used equations of state are:

$$w = \begin{cases} 0 & \text{dust, matter-dominated} \\ \frac{1}{3} & \text{radiation, radiation-dominated} \\ 1 & \text{stiff matter} \\ -1 & \text{dark energy} \end{cases} \tag{1.39}$$

The cosmological field equations of general relativity are

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1.40}$$

where G_{ab} is the Einstein tensor.

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{1.41}$$

and Λ is the cosmological constant.

Consider the FLRW metric in the form

$$ds^2 = -dt^2 + \frac{dx^2 + dy^2 + dz^2}{(1 + \frac{k}{4}(x^2 + y^2 + z^2))^2} \tag{1.42}$$

The Ricci tensor components are given by

$$R_t^t = 3\frac{\ddot{a}}{a} \tag{1.43}$$

$$R_x^x = R_y^y = R_z^z = \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} \tag{1.44}$$

$$R_a^b = 0 \quad \text{if} \quad a \neq b \tag{1.45}$$

$$R = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a} + 6\frac{k}{a^2} \tag{1.46}$$

where the dot means derivative with respect to cosmological time t. This yields the following non-vanishing Einstein tensor components

$$G_t^t = -3\frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2} \tag{1.47}$$

$$G_x^x = G_y^y = G_z^z = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} \tag{1.48}$$

while the components of the stress-energy tensor are

$$T^a_b = \mathrm{diag}(-\rho, P, P, P) \tag{1.49}$$

Hence the cosmological field equations are

$$(tt) \qquad -3\frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2} + \Lambda = -8\pi\rho \tag{1.50}$$

$$(\text{spatial}) \qquad -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = 8\pi P \tag{1.51}$$

Rearranging the terms slightly gives

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} \tag{1.52}$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = 8\pi P - \Lambda \tag{1.53}$$

and if we finally add up both equations, we arrive at

$$-2\frac{\ddot{a}}{a} = \frac{8\pi}{3}(\rho + 3P) - \frac{2}{3}\Lambda \tag{1.54}$$

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3P) - \frac{\Lambda}{3}. \tag{1.55}$$

Therefore we can alternatively write the field equations as and we finally find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.56}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{1.57}$$

These are the so called Friedmann equations which are the starting point of analysing cosmological solutions of the Einstein field equations.

The twice contracted Bianchi identities yield energy-momentum conservation:

$$\nabla_a T^{ab} = 0 \tag{1.58}$$

These equations are not independent since they are implied by the field equations.

To see this explicitely, consider the time derivative of (1.56)

$$2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{8\pi}{3}\dot{\rho} + 2\frac{k}{a^2}\frac{\dot{a}}{a} \tag{1.59}$$

From the field equation we also find

$$8\pi(\rho + P) = 2\frac{k}{a^2} - 2\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \tag{1.60}$$

$$8\pi(\rho + P)\frac{\dot{a}}{a} = 2\frac{k}{a^2}\frac{\dot{a}}{a} - 2\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \tag{1.61}$$

which then yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{1.62}$$

which is the energy-momentum conservation equation in cosmology.

1.7 Cosmological solutions of the field equations

1.7.1 Friedman solutions – matter-dominated cosmos

A matter dominated universe is described by the following energy-momentum tensor

$$T_{ab} = \rho u_a u_b. \tag{1.63}$$

Neglecting for the moment the cosmological constant, i.e. $\Lambda = 0$, the field equations become

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{1.64}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho \tag{1.65}$$

which the conservation equation reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0. \tag{1.66}$$

The conservation equation can be integrated

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \Rightarrow \quad (\log\rho)^{\cdot} = -3(\log a)^{\cdot} \tag{1.67}$$

Hence, the scale factor and the energy density are related by

$$\log\rho = -3\log a + C \quad \Rightarrow \quad \rho = \rho_0\frac{1}{a^3} \tag{1.68}$$

Zero spatial curvature $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 \frac{1}{a^3} \quad \Rightarrow \quad \dot{a}^2 = \frac{8\pi}{3}\rho_0 \frac{1}{a} \tag{1.69}$$

Taking the square root and separating variables yields

$$dt = \pm \frac{1}{\sqrt{\frac{8\pi}{3}\rho_0}} \sqrt{a}\, da \tag{1.70}$$

which is integrated straightforwardly to give

$$(t - t_0) = \pm \frac{1}{\sqrt{8\pi\rho_0/3}} \,(2/3)\, a^{3/2} \tag{1.71}$$

and therefore the scale factor is given by

$$a(t) = \pm(6\pi\rho_0)^{1/3}\,(t - t_0)^{3/2}. \tag{1.72}$$

Let us now assume the universe started with zero volume $a(t_0) = 0$ and set $t_0 = 0$. We then find

$$a(t) \sim t^{2/3} \qquad \frac{\dot{a}}{a} = \frac{2}{3}\frac{1}{t} \tag{1.73}$$

$$\rho(t) \sim \frac{1}{a^3} \sim \frac{1}{t^2} \tag{1.74}$$

Hence, the energy density decreases as a function of time, however, it diverges as $t \to 0$. This corresponds to big bang. Note that the divergent energy density, which corresponds to divergent curvature, is a genuine feature of cosmological models. This solution is called the Einstein-de Sitter cosmos. There exists a particle horizon at $r = 3t/a$.

Positive spatial curvature $k = +1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 \frac{1}{a^3} - \frac{1}{a^2} \tag{1.75}$$

Separation of variables gives

$$\pm dt = \frac{da}{\sqrt{\frac{8\pi}{3}\rho_0 \frac{1}{a_0} - 1}} \tag{1.76}$$

which after integration yields

$$\pm t = \frac{8\pi}{3}\rho_0 \arctan\Big(1/\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{a} - 1}\Big) - \sqrt{a}\sqrt{\frac{8\pi}{3}\rho_0 - a} \tag{1.77}$$

which one cannot solve explicitly for $a(t)$. However, it is possible to derive a solution in parametrised form.

Let us define

$$1/\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{a} - 1} = \tan u \quad \Leftrightarrow \quad a = \frac{8\pi}{3}\rho_0 \sin^2 u \tag{1.78}$$

which leads to

$$\pm t = \frac{8\pi}{3}\rho_0(u - \sin u \cos u) \tag{1.79}$$

Next, taking into account

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u) \tag{1.80}$$

$$\sin u \cos u = \frac{1}{2}\sin 2u \tag{1.81}$$

and finally rescaling $u \to u/2$, one arrives at the standard parametrisation of a cycloid

$$a = \frac{4\pi}{3}\rho_0(1 - \cos u) \tag{1.82}$$

$$\pm t = \frac{4\pi}{3}\rho_0(u - \sin u) \tag{1.83}$$

At $t = 0 (u = 0)$, the solution satisfies $a(0) = 0$. The universe expands until reaching the maximal value

$$a_{\max} = \frac{8\pi}{3}\rho_0 \tag{1.84}$$

at $t_{\max} = 4\pi^2\rho_0/3$ where $u = \pi$, after which it contracts again to $a = 0$ at $t = 8\pi^2/(3\rho_0)$. This is often called big crunch (opposed to big bang).

Negative spatial curvature $k = -1$ In that case we need to solve

$$\pm dt = \frac{da}{\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{a} + \frac{1}{a^2}}} \tag{1.85}$$

from which we find

$$\pm(t - t_0) = \sqrt{a}\sqrt{a + \frac{8\pi}{3}\rho_0} - \frac{8\pi}{3}\rho_0 \log\left[3\sqrt{a} + \sqrt{g_a + 24\pi\rho_0}\right] \tag{1.86}$$

The constant of integration t_0 is fixed by the condition

$$a(t_0 = 0) = 0. \tag{1.87}$$

Introducing, as before, a parameter u,

$$a = \frac{8\pi}{3}\rho_0 \sinh^2 u \tag{1.88}$$

$$\pm t = \frac{4\pi}{3}\rho_0(\sinh 2u - 2u) \tag{1.89}$$

which one can rewrite

$$a = \frac{4\pi}{3}\rho_0(\cosh 2u - 1) \tag{1.90}$$

$$\pm t = \frac{4\pi}{3}\rho_0(\sinh 2u - 2u) \tag{1.91}$$

This solution expands, as $t \to \infty$ it expands asymptotically like $a(t) \sim t$.

use.eps

use.eps

1.7.2 Friedman solutions - radiation dominated cosmos

A sufficiently hot and dense universe can no longer be described by a matter-dominated model. Radiation becomes the main source of gravitational field and this is described by the equation of state

$$P = \frac{1}{3}\rho \tag{1.92}$$

Therfore, the conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = \dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \tag{1.93}$$

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \quad \Rightarrow \quad (\log\rho)^{\cdot} = (-4\log a)^{\cdot} \tag{1.94}$$

which can be integrated to give

$$\rho = \rho_0 \frac{1}{a^4} \tag{1.95}$$

Consider the case with zero spatial curvature $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0\frac{1}{a^4} \Rightarrow \dot{a}^2 = \frac{8\pi}{3}\rho_0\frac{1}{a^2} \tag{1.96}$$

Separation of variables yields

$$\pm dt = \frac{a\,da}{\sqrt{\frac{8\pi}{3}\rho_0}} \quad \Rightarrow \quad \pm(t - t_0) = \frac{1}{2}\frac{1}{\sqrt{\frac{8\pi}{3}\rho_0}}a^2 \tag{1.97}$$

choosing $a(t_0) = 0$ and $t_0 = 0$ we find

$$a(t) = \sqrt{2}(8\pi\rho_0/3)^{1/4}\, t^{1/2} \qquad \frac{\dot{a}}{a} = \frac{1}{2}\frac{1}{t} \tag{1.98}$$

and

$$\rho(t) \sim \frac{1}{t^2} \tag{1.99}$$

which equals the functional form of the energy density of the matter dominated universe. The particle horizon is at $r = 2t/a$.

1.8 Friedman-Lemaître solutions

Let us now analyse solutions with the cosmological constant Λ present in the field equations.

1.8.1 The Einstein static universe

Consider the field equations with $k = +1$ and $\Lambda \neq 0$. Is it possible to find a static solution where $a = a_E =$ constant and $\dot{a} = \ddot{a} = 0$?

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.100}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{1.101}$$

Constancy of the scale factor implies

$$0 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{1}{a_E^2} \tag{1.102}$$

$$0 = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{1.103}$$

which can be used to determine a_E and the cosmological constant

$$\frac{1}{a_E^2} = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} \tag{1.104}$$

$$\Lambda = 4\pi(\rho + 3p) \tag{1.105}$$

This solution is the Einstein static universe. It is unstable with respect to small perturbations $a_E \to a_E + \delta a$, $\rho \to \rho + \delta\rho$.

1.8.2 The de Sitter solution

Let us try to solve the field equation in the absence of matter $\rho = p = 0$ with $\Lambda \neq 0$ and $k = 0$. The resulting field equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \tag{1.106}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} \tag{1.107}$$

This is equivalent to

$$(\log a)^{\cdot} = \sqrt{\frac{\Lambda}{3}} \tag{1.108}$$

$$a(t) = \exp(\sqrt{\Lambda/3}t) \tag{1.109}$$

with $a(t = 0) = 1$. Therefore, the resulting de Sitter line-element is given by

$$ds^2 = -dt^2 + e^{2\sqrt{\Lambda/3}t}(dx^2 + dy^2 + dz^2) \tag{1.110}$$

The de Sitter solution has some interesting properties that we will discuss. It seems that the de Sitter metric is time-dependent, in fact, it is static!

Consider the time translation $t \to t = t' + t_0$.

$$ds^2 \to ds'^2 = -dt'^2 + e^{2\sqrt{\Lambda/3}t'}e^{2\sqrt{\Lambda/3}t_0}(dx^2 + dy^2 + dz^2) \tag{1.111}$$

and rescale the spatial coordinates $x' = e^{\sqrt{\Lambda/3}t_0}\,x$ etc, which results in

$$ds'^2 = -dt'^2 + e^{2\sqrt{\frac{\Lambda}{3}}t'}(dx^2 + dy^2 + dz^2). \tag{1.112}$$

Hence this metric is form invariant under time translations and therefore static.

The coordinates (t, x, y, z) do not cover the complete manifold. To find a better coordinate system, let us analyse the four dimensional hyperboloid

$\mathbb{H}^4$ embedded in $\mathbb{R}^5$ with Lorentzian signature

$$v = \sqrt{\frac{3}{\Lambda}} \sinh \sqrt{\frac{\Lambda}{3}} t \tag{1.113}$$

$$w = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t \cos\chi \tag{1.114}$$

$$x = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t \sin\chi \cos\theta \tag{1.115}$$

$$y = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t \sin\chi \sin\theta \cos\phi \tag{1.116}$$

$$z = \sqrt{\frac{3}{\Lambda}} \cosh \sqrt{\frac{\Lambda}{3}} t \sin\chi \sin\theta \sin\phi \tag{1.117}$$

with $-\infty < v, w, x, y, z < \infty$ and $-v^2 + w^2 + x^2 + y^2 + z^2 = \frac{3}{\Lambda}$.

The induced metric on $\mathbb{R}^5$

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2 \tag{1.118}$$

becomes

$$ds^2 = -dt^2 + \frac{3}{\Lambda} \cosh^2(\sqrt{\Lambda/3}t) \left[d\chi^2 + \sin^2\chi d\Omega^2\right] \tag{1.119}$$

The original de Sitter metric is obtained by introducing new coordinates

$$T = \sqrt{\frac{3}{\Lambda}} \log \left[\frac{v+w}{\sqrt{3/\Lambda}}\right] \tag{1.120}$$

$$X = \sqrt{\frac{3}{\Lambda}} \frac{x}{v+w}, \quad Y = \dots \tag{1.121}$$

which gives exactly

$$ds^2 = -dT^2 + e^{2\sqrt{\Lambda/3}T} \left[dX^2 + dY^2 + dZ^2\right] \tag{1.122}$$

which only covers the $v + w > 0$ region.

1.8.3 Qualitative analysis of FL solutions

In the matter dominated epoch we have

$$\rho_m \sim \frac{1}{a^3} \quad \Rightarrow \quad K_m = \frac{8\pi}{3} \rho_m a^3 = \text{const.} \tag{1.123}$$

while for radiation we find

$$\rho_r \sim \frac{1}{a^4} \quad \Rightarrow \quad K_r = \frac{8\pi}{3}\rho_r a^4 = \text{const.} \tag{1.124}$$

Let us write the total energy density of the universe as

$$\rho_{\text{total}} = \rho_m + \rho_r \tag{1.125}$$

and assume there are no interactions between radiation and matter. Since we already used the conservation equations, the remaining field equation takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_{tot} + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{1.126}$$

$$\Leftrightarrow \quad \dot{a}^2 = \frac{8\pi}{3}\rho_r a^2 + \frac{8\pi}{3}\rho_m a^2 + \frac{\Lambda}{3}a^2 - k \tag{1.127}$$

$$\Leftrightarrow \quad \dot{a}^2 = \frac{K_r}{a^2} + \frac{K_m}{a} + \frac{\Lambda}{3}a^2 - k \tag{1.128}$$

$$\Leftrightarrow \quad \dot{a}^2 - \frac{K_r}{a^2} - \frac{K_m}{a} - \frac{\Lambda}{3}a^2 = -k \tag{1.129}$$

which is of the form of a one-dimensional mechanical system

$$\dot{a}^2 + V_{\text{eff}}(a) = -k \tag{1.130}$$

with

$$V_{\text{eff}}(a) = -\frac{K_r}{a^2} - \frac{K_m}{a} - \frac{\Lambda}{3}a^2 \tag{1.131}$$

This equation of motion describes the dynamics of our idealised universe.

Without radiation and without cosmological constant we find

$$\frac{M}{2}\left(\frac{da}{dt}\right)^2 - \frac{GM^2}{a} = \text{const.} = -\frac{kM}{2} \tag{1.132}$$

where we used $K_m = 8\pi\rho_m a^3/3 = 2M$. This equation we can read as: *kinetic energy + potential energy = total energy*

In the analogous Newtonian two-body problem we have

$$m\frac{\dot{r}}{2} + V_{\text{eff}}(r) = \text{const.} \tag{1.133}$$

Therefore we an conclude that $k = 1$ corresponds to closed orbits (ellipses), $k = -1$ to unbound orbits (hyperbola) while $k = 0$ is the limiting case (parabola).

If applied to the universe we find

- $k = 1$ universe expands and later contracts (bound system).
- $k = 0$ expansion velocity asymptotically approaches zero.
- $k = -1$ kinetic energy dominates, expansion never stops.

With Λ the effective potential (with $k < 0$) is shown in Fig.

`use.eps`

2 Cosmological parameters and observable quantities

2.1 Cosmological parameters

The FLRW metric contains the unknown function $a(t)$. A finite number of observations will never fix this function completely, however, we can try to approximate it. We introduce the functions

$$H := \frac{\dot{a}}{a} \tag{2.1}$$

$$q_0 := -\frac{a\ddot{a}}{\dot{a}^2} = -\frac{1}{H^2}\frac{\ddot{a}}{a} \tag{2.2}$$

$$q_n := (-1)^{n+1}\frac{a^{(n+2)}}{H^{(n+2)}a} \tag{2.3}$$

In physical units we have

$$H = \frac{1}{a}\frac{da}{dt}, \quad [H] = (\text{time})^{-1}, \quad [H/c] = (\text{length})^{-1} \tag{2.4}$$

and the q_n are dimensionless parameters.

Given a fixed epoch t_0, the quantities $H(t_0)$ and $q_n(t_0)$ are in principle observable constants. Hence, we can approximate $a(t)$ by

$$a(t) = a(t_0)\left[1 + x - \frac{q_0}{2}x^2 + \frac{q_1}{3!}x^3 - ....\right] \tag{2.5}$$

where we denoted

$$x = (t - t_0)H(t_0). \tag{2.6}$$

H is the Hubble function, often called Hubble constant when it refers to its currently observed value. H measures/describes the relative expansion rate. H and is usually given in units of km/s/Mpc.

2.2 Redshift

Electromagnetic signals are describes by null geodesics. Consider the line-element in the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\Omega^2\right] \tag{2.7}$$

and consider radial geodesics $\theta = \theta_0$ and $\phi = \phi_0$.

Hence, $ds^2 = 0$ yields

$$\int_{r1}^{r} \frac{d\rho}{\sqrt{1-k\rho^2}} = -\int_{t_1}^{t} \frac{dt'}{a(t')} \tag{2.8}$$

and let us define

$$\sigma(r) := \int_{0}^{r} \frac{d\rho}{\sqrt{1-k\rho^2}} \geq 0 \tag{2.9}$$

Assuming a second signal from the origin at time $t + \delta t$ which reaches r at $t + \delta t_1$, we find

$$\int_{t+\delta t}^{t_1+\delta t_1} \frac{dt'}{a(t')} = \sigma(r) \tag{2.10}$$

Taking the difference and using $\delta t_1 \ll t_1$ and $\delta t \ll t$ yields

$$\frac{\delta t}{a(t)} - \frac{\delta t_1}{a(t_1)} = 0 + O((\delta t)^2, (\delta t_1)^2) \tag{2.11}$$

Let us interpret δt and δt_1 as the time intervals between two maxima of the signal's wave. Then we may write

$$\delta t = \lambda, \delta t_1 = \lambda_1 \tag{2.12}$$

and the redshift is defined by

$$z := \frac{\lambda_1 - \lambda}{\lambda} = \frac{a(t_1)}{a(t)} - 1 \tag{2.13}$$

$$1 + z = \frac{a(t_1)}{a(t)} \tag{2.14}$$

2.3 Luminosity distance

Assume a radiating point-like source at $r = 0$, $\theta = \theta_0$ and $\phi = \phi_0$ with absolute luminosity L. It radiates during an interval δt the wave length interval $\delta\lambda$. Consider an observer at r_1 and time t_1 who measures δt_1 and $\delta\lambda_1$ as energy per area, time interval and wave length interval

$$L\delta t\delta\lambda = L_{\text{abs}}\delta t_1\delta\lambda_1 F_1 \tag{2.15}$$

We define the (integrated) eigen distance

$$d(t) := a(t)\sigma(r) \tag{2.16}$$

At time t_1 the radiation is spread out

$$F_1 = 4\pi d(t_1)^2 = 4\pi a^2(t_1)\sigma(r_1)^2 \tag{2.17}$$

For a sphere of radius r_1 the eigen distance is distance between the source and the surface of the sphere, hence the radius r_1

$$F_1 = 4\pi a^2(t_1)r_1^2 \tag{2.18}$$

which gives

$$L = L_{\text{obs}}(1+z)^2 4\pi a^2(t_1)r_1^2 \tag{2.19}$$

This suggests the definition of the luminosity distance (in analogy with Euclidean geometry)

$$L = L_{\text{obs}} 4\pi d_L^2 \tag{2.20}$$

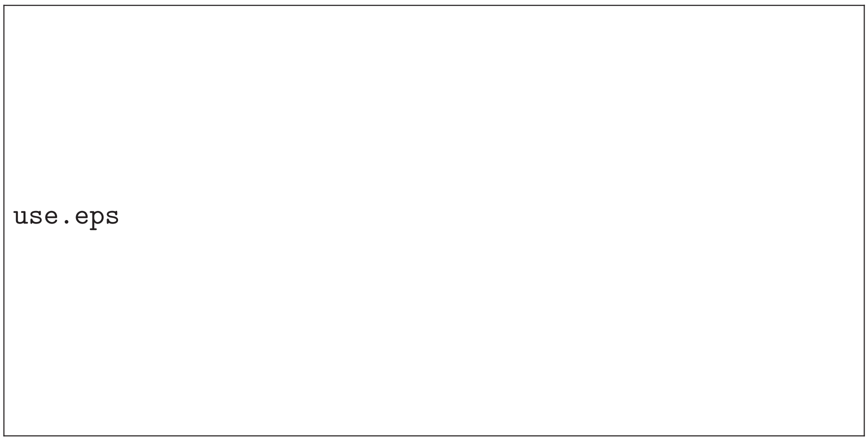
where

$$d_L = (1+z)a(t_1)r_1 \tag{2.21}$$

Sometimes the *'Euclidean'* distance is used

$$d_c = a(t_1)r_1 = \frac{d_L}{(1+z)} \tag{2.22}$$

2.4 Angular diameter distance



We can express the distance δ as follows:

$$\frac{\delta(t_1)}{2d(t)} = \tan\frac{\theta_1}{2} \simeq \frac{\theta_1}{2} \tag{2.23}$$

with $\theta \ll 1$ and therefore

$$\delta(t_1) = d(t)\theta_1 = a(t)r\theta_1 = \frac{a(t_1)r}{1+z}\theta_1. \tag{2.24}$$

In analogy to Euclidean geometry we define

$$\delta(t_1) = d_A\theta_1 \tag{2.25}$$

$$d_A = \frac{a(t_1)r}{1+z} \tag{2.26}$$

and therefore we find

$$d_L = (1+z)^2 d_A \tag{2.27}$$

The definitions of d_L and d_A contain coordinate dependent quantities. In the following we express them in terms of observables.

2.5 Luminosity distance redshift relation

In galactic astronomy the 'standard distance' is 10pc. The apparent magnitude m of an object is a measure of its brightness as seen on Earth. The absolute magnitude M is the apparent magnitude if the object were at a standard luminosity distance

$$m - M = 5\log_{10}\Big(\frac{d[\mathrm{pc}]}{10pc}\Big) \tag{2.28}$$

For nearby objects the luminosity distance is almost identical to the physical/real distance. Spacetime is nearly Euclidean within our galaxy and hence the effects are curvature are nearly negligible.

The redshift is z was defined by

$$z = \frac{a(t_1)}{a(t)} - 1 \tag{2.29}$$

Let us express our approximation of $a(t)$ using the redshift z which gives

$$z = (t_1 - t)H(t_1) + (t_1 - t)^2 H^2(t_1)(1 + \frac{q_0(t_1)}{2}) + \mathit{0}(H(t_1^3)) \tag{2.30}$$

Let us solve this equation for the Hubble function

$$(t_1 - t)H(t_1) = z - (1 + \frac{q_0(t_1}{2})z^2 + O(z^3). \qquad (2.31)$$

For radial null geodesics with $r \geq 0$ we defined

$$\sigma(r) = \sigma(r_1) - \int_{t_1}^{t} \frac{dt'}{a(t')} \qquad (2.32)$$

so we can Taylor expand $\sigma(r)$ such that

$$\sigma(r) = \sigma(r_1) + \left.\frac{\partial \sigma}{\partial t}\right|_{t_1} (t - t_1) + \frac{1}{2} \left.\frac{\partial^2 \sigma}{\partial t^2}\right|_{t_1} (t - t_1)^2 + O(t - t_1)^3 \qquad (2.33)$$

Now, using the definition of $\sigma(r)$, we find

$$\frac{\partial \sigma}{\partial t} = \frac{1}{a}, \qquad \frac{\partial^2 \sigma}{\partial t^2} = -\frac{\dot{a}}{a^2} = -\frac{H}{a} \qquad (2.34)$$

and hence we find

$$a(t_1)\sigma(r) = a(t_1)\sigma(t_1) + (t - t_1) - \frac{1}{2}H(t_1)(t - t_1)^2 + O(t - t_1)^3 \qquad (2.35)$$

If the source is located at $r = 0$, we have $\sigma(r) = 0$ and hence

$$a(t_1)\sigma(r_1) = \frac{z}{H(t_1)} - \frac{1}{2}(1 + q_0(t_1))\frac{z^2}{H(t_1)} + O(z^3) \qquad (2.36)$$

Recall the definition of the luminosity distance

$$d_L = (1 + z)a(t_1)\sigma(r_1)\frac{r_1}{\sigma(r_1)} \qquad (2.37)$$

Note that

$$\frac{\sigma(r_1)}{r_1} = 1 - \frac{1}{6}k(r_1)^2 + O(r_1^3) \qquad \forall k \qquad (2.38)$$

Therefore, using the approximation for $\sigma(r)$ we find

$$r_1 = \frac{z}{a(t_1 H(t_1)} \left[1 - \frac{1}{2}(1 + q_0(t_1))z\right] + O(z^3)$$
$$\frac{\sigma(r_1)}{r_1} = 1 - \frac{1}{6}k\frac{z^2}{a(t_1)^2 H(t_1)^2} + O(z^3) \qquad (2.39)$$

Finally, we can express the luminosity distance in terms of observable quantities and arrive at

$$d_L = \frac{z}{H(t_1)}\left[1 + \frac{1}{2}(1 - q_0(t_1))z\right] + O(z^3) \tag{2.40}$$

In the lowest order in the redshift z we obtain

$$z \simeq H(t_1) d_L \tag{2.41}$$

which is Hubble's law.

The luminosity redshift relation is obtained by inserting d_L into the magnitudes we find up to order z

$$m = M - 5 - 5\log_{10} H(t_1) + 5\log_{10} z + \frac{5}{2}\log_{10} e(1 - q_0(t_1)z \tag{2.42}$$

in which we used $1/\log 10 = \log_{10} e$.

Example 2.1. Angular diameter distance and luminosity distance redshift relation of a flat ($k = 0$) matter dominated $P = 0$ universe with $\Lambda = 0$. Start with

$$d_A = (1 + z)^{-1} a(t_1) r_1 \tag{2.43}$$

using that $a(t) = a_0 t^{2/3}$ and $H = 2/3\, 1/t$

$$r_1 = \sigma(r_1) = \int_t^{t_1} \frac{dt'}{a(t')} = a_0^{-1}\int_t^{t_1} t^{-2/3} dt' \tag{2.44}$$

$$= 3a_0^{-1}\left[(t')^{1/3}\right]_t^{t_1} = 3a_0^{-1}(t_1^{1/3} - t^{1/3}) = 3a_0^{-1} t_1^{1/3}\left(1 - \frac{t^{1/3}}{t_1^{1/3}}\right) \tag{2.45}$$

Therefore, we first of all find

$$d_A = (1 + z)^{-1}(a_0 t_1^{2/3})(3a_0^{-1} t_1^{1/3})(1 - \frac{t^{1/3}}{t_1^{1/3}}) \tag{2.46}$$

Now we use that

$$\left(\frac{t}{t_1}\right)^{2/3} = \frac{a(t)}{a(t_1)} = (1 + z)^{-1} \tag{2.47}$$

and therefore we have

$$d_A = (1 + z)^{-1} 3t_1[1 - (1 + z)^{-1/2}] \tag{2.48}$$

Finally we arrive at

$$d_A = \frac{2}{H_1}[(1+z)^{-1} - (1+z)^{-3/2}] \tag{2.49}$$

This also fixes the luminosity distance

$$d_L = \frac{2}{H_1}[(1+z) - \sqrt{1+z}] \tag{2.50}$$

expressed in terms of the redshift.

2.6 Density parameters and field equations

The field equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{2.51}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{2.52}$$

Using the Hubble parameter in the first equation

$$H^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \tag{2.53}$$

followed by division by H^2 gives

$$1 = \frac{8\pi}{3H^2}\rho + \frac{\Lambda}{3H^2} - \frac{k}{a^2H^2} \tag{2.54}$$

The one of the left-hand side tell us that all the terms on the right-hand must be dimensionless as well, and that it may be useful to introduce dimensionless parameters.

Let us introduce density parameters as follows

$$\Omega = \frac{8\pi}{3H^2}\rho = \frac{\rho}{\rho_c} \tag{2.55}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \tag{2.56}$$

where ρ_c is the critical density defined by

$$\rho_c = \frac{3H^2}{8\pi} \tag{2.57}$$

which is a function of time.

This results in the following form of the Friedman equation

$$1 = \Omega + \Omega_\Lambda - \frac{k}{a^2 H^2} \tag{2.58}$$

Let $\Omega_{\text{total}} = \Omega + \Omega_\Lambda$ be the total density parameter of the universe, then we have

$$\Omega_{\text{total}} - 1 = \frac{k}{a^2 H^2}. \tag{2.59}$$

In general Ω_{total} is a time dependent function, however, if $k = 0$ (spatially flat constant time hypersurfaces) $\Omega_{total} = 1$, it retains this value forever.

Note that this equation in principle allows to determine the sign of k by measuring the matter content of the universe.

Example 2.2. Consider the matter dominated cosmos with $P = 0$ and $\Lambda = 0$. For the Hubble parameter, the deceleration parameter and the density parameter we find

$$H = \frac{2}{3}\frac{1}{t}, \qquad q_0 = \frac{1}{2}, \qquad \Omega = 1. \tag{2.60}$$

2.7 Contents of the universe

For small redshifts $z \ll 1$ Hubble's law determines the relative velocity of a pair of nearby comoving observers

$$z = H dr \tag{2.61}$$

If distances of galaxies were accurately known one could find the present value H_0. The subscript 0 always refers to present day values in cosmology. There are many uncertainties in H_0. It is usually parametrised by h defined by

$$H_0 = 100\, h \frac{\text{km}}{\text{s Mpc}} \simeq \frac{h}{3000} \frac{1}{\text{Mpc}} \tag{2.62}$$

Observations suggest

$$0.5 < h < 0.8 \tag{2.63}$$

with $h \simeq 0.7$ being the most recent.

The inverse Hubble parameter yields a time scale, the Hubble time

$$H_0^{-1} \simeq 9.78 h^{-1} \text{Gyr} \tag{2.64}$$

and the Hubble distance

$$cH_0^{-1} \simeq 2998h^{-1}\text{Mpc} \tag{2.65}$$

Most of the mass of the universe does not interact with radiation (neither emits nor absorbs), it appears to be dark as opposed to luminous.

Various observations suggest

$$0.3 < \Omega_0 < 0.5 \tag{2.66}$$

The value of the cosmological constant Λ can also be found observationally, for example by observing distant type Ia supernovae.

For flat cosmological models $k = 0$ the best fit to data yields

$$\Omega_0 \simeq 0.3 \tag{2.67}$$

with the cosmological constant Ω_Λ comprising the remainder

$$\Omega_{\text{total}} = \Omega_0 + \Omega_\Lambda = 1 \tag{2.68}$$

Gravitational lensing gives

$$\Omega_\Lambda < 0.74 \tag{2.69}$$

at 2σ confidence. Also large-scale structure models support this limit and hence we may confidently assume

$$0.3 < \Omega_0 < 0.4 \tag{2.70}$$

Ordinary matter in cosmology is referred to as baryons since protons and neutrons account for all its density. The baryonic density is denoted by Ω_b. Theoretical calculations of nucleosynthesis yield the following limit on the baryon density

$$0.010 \leq \Omega_b h^2 \leq 0.022 \tag{2.71}$$

Clusters of galaxies observations may constrain the ratio $\frac{\Omega_b}{\Omega_0}$. It is assumed that these clusters represent the Universe as a whole

$$\frac{\Omega_b}{\Omega_0} = 0.14^{+0.08}_{-0.04} \left(\frac{h}{0.5}\right)^{-3/2} \tag{2.72}$$

Together with the nucleosynthesis bounds one can constrain Ω_0.

The dark matter can either be in the form of baryonic (for instance cold gas) matter or non-baryonic dark matter. There are many dark matter candidates ranging from massive neutrinos, axions, to lightest supersymmetric particles or geometrical generalisations.

The present value of the critical energy density is

$$\rho_{c,0} = 1.88h^2 \times 10^{-29}\text{g/cm}^3 \tag{2.73}$$

(Recall that the proton mass $m_p = 1.67x10^{-24}g$ which means that the critical density today is about one proton per cubic meter).

2.8 Temperature

We only state a few basic facts about the temperature of the universe and how it is related the scale factor.

2.8.1 Radiation dominated universe

Liouville's theorem of classical mechanics can be applied to the phase space of the cosmic matter. This allows to relate the expansion parameter to the temperature

$$\frac{T(t)}{T(t_1)} = \frac{a(t_1)}{a(t)} \tag{2.74}$$

which also relates temperatures to redshifts

$$\frac{T(t)}{T(t_1)} = 1 + z \tag{2.75}$$

2.8.2 Matter dominated universe

In the matter dominated epoch one finds instead

$$\frac{T(t)}{T(t_1)} = \frac{a(t_1)^2}{a(t)^2} \tag{2.76}$$

or

$$\frac{T(t)}{T(t_1)} = (1 + z)^2 \tag{2.77}$$

At a temperature of about $0.31 \sim 3.6 \times 10^3$K photons decouple from matter and hydrogen is formed, the so called decoupling or recombination.

(Note the misnomer: the objects were never previously combined.) Hence, the redshift is

$$z_{rec} = \frac{T(t_{rec})}{T(t_1)} - 1 \tag{2.78}$$

Let us interpret the cosmic microwave background radiation as the cooled gas of photons. Since we observe $T(t_1) \sim 2.73\text{K}$. This gives $z_{rec} \simeq 1330$.

Let us compare this temperature of ($T(t_1) \sim 2.73\text{K}$) the photon gas with the temperature of the matter

$$T(t_1) = \frac{3.6 \times 10^3\text{K}}{(1 + z_{rec})^2} \simeq 2 \times 10^{-3}\text{K}. \tag{2.79}$$

Hence, the matter component cools down much faster than the radiation.

2.9 History of the Universe

Table in progress.

3 Inflation

3.1 Shortcomings of standard cosmology

3.1.1 Flatness problem

In terms of the density parameter Ω, the time-time component of the cosmological field equations is

$$\Omega - 1 = \frac{k}{a^2 H^2} \tag{3.1}$$

If the constant time hypersurfaces are flat, $k = 0$, then $\Omega = 1$ and it remains so far all time.

If $k \neq 0$, the density parameter evolves. In a nearly flat universe that is matter dominated, we derived

$$a \sim t^{\frac{2}{4}}, \quad H \sim \frac{1}{t} \quad \rightarrow \quad aH \sim t^{-\frac{1}{3}} \tag{3.2}$$

while for the radiation dominated epoch we obtained

$$a \sim t^{\frac{1}{2}}, \quad H \sim \frac{1}{t} \quad \rightarrow \quad aH \sim t^{-\frac{1}{2}} \tag{3.3}$$

and therefore

$$|\Omega - 1| \propto \begin{cases} t & \text{radiation dominated} \\ t^{2/3} & \text{matter dominated} \end{cases} \tag{3.4}$$

The flatness problem is that $1/(aH)$ is in general an increasing function. We observe Ω_0 of the order unity and therefore Ω was very close to unity at early times. For instance at nucleosynthesis we must require

$$|\Omega(t_{\text{nucleosynthesis}}) - 1| < 10^{-16} \tag{3.5}$$

in order to obtain our present universe.

Such finely-tuned initial conditions are very problematic as such conditions are unlikely.

3.1.2 Horizon problem

We discussed the possible presence of particle horizons depending on the convergence of the integral

$$\eta = \int \frac{dt'}{a(t')} \tag{3.6}$$

In the radiation dominated epoch and also in the matter dominated epoch there are particle horizons. Hence, there exist regions in spacetime that cannot interact.

On the other hand, the cosmic microwave background is (nearly) homogeneous. However, these similarly looking regions cannot have interacted before recombination. This suggests that the homogeneity we observe must have been encoded in the initial conditions.

Homogeneity and isotropy problem

We discussed that the large scale homogeneity and isotropy might be encoded in the initial conditions. One of the key questions is whether there exists a theoretical model that can explain the inhomogeneity.

In particular these inhomogeneities should have been much longer than the corresponding horizon scale. As before, this requires fine tuned initial conditions.

3.2 Accelerated expansion – inflation

A cosmological epoch where the universe is accelerating is called inflation. Formally we can define inflation by

$$\ddot{a} > 0. \tag{3.7}$$

This condition can be rewritten more physically, recall the definition of the Hubble parameter

$$H = \frac{\dot{a}}{a}, \qquad \dot{a} = Ha \tag{3.8}$$

Let us consider the following quantity

$$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) = \frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = -\frac{1}{(\dot{a})^2}\ddot{a} < 0 \tag{3.9}$$

and therefore we could also define inflation by

$$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0 \tag{3.10}$$

The quantity $\frac{H^{-1}}{a}$ is the comoving Hubble length. This is an important length scale since it determines if two regions can communicate now. If they are separated by distances greater than $\frac{H^{-1}}{a}$ they cannot communicate.

Note that the particle horizon (sometimes comoving horizon) separating two regions means that they never could have communicated with one another.

In principle this allows the following possibility: The length scale of the particle horizon could be much larger than the comoving Hubble length today; particles cannot communicate today but were in causal contact early on.

Actually, during inflation the observable universe becomes smaller. The characteristic scale occupies a smaller coordinate size.

Recall a particular linear combination of the cosmological field equations with $\Lambda = 0$. We derived

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3P) \tag{3.11}$$

Yet another way of defining an accelerated expansion of the universe $\ddot{a} > 0$ is therefore

$$\rho + 3P < 0 \tag{3.12}$$

Since the energy density ρ is assumed to be positive, acceleration requires a negative pressure. If we assume equation of state of the form

$$P = w\rho \tag{3.13}$$

we find

$$\rho + 3w\rho = \rho(1 + 3w) < 0 \quad \Leftrightarrow \quad w < -\frac{1}{3} \tag{3.14}$$

3.3 Solving the problems

Flatness problem The condition for inflation is

$$\frac{d}{dt}\left(\frac{1}{Ha}\right) < 0 \tag{3.15}$$

and hence $|\Omega - 1|$ is driven towards zero rather than away from it.

Horizon problem The horizon problem is solved because of the reduction of comoving Hubble length during inflation.

3.4 Scalar fields in cosmology

Spin 0 particles or scalar particles are described by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) \tag{3.16}$$

which leads to the following stress-energy tensor

$$T_{ab} = \nabla_a \phi \nabla_b \phi + g_{ab} \mathcal{L} \tag{3.17}$$

The covariant derivate when acting on the scalar is simply the partial derivative and therefore we can write $\partial_a \phi$ instead. In cosmology we assume that all fields are time dependent only and hence $\phi = \phi(t)$. Let us furthermore assume spatially flat hypersurfaces, then the metric reads

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{3.18}$$

$$g_{ab} = \mathrm{diag}(-1, a^2, a^2, a^2) \tag{3.19}$$

In that particular case, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - V(\phi) \tag{3.20}$$

compare with the harmonic oscillator.

Let us now compute the components of the energy-momentum tensor.

$$T_{tt} = \dot{\phi}^2 - \mathcal{L} = \frac{1}{2} \dot{\phi}^2 + V(\phi) \tag{3.21}$$

$$T_{xx} = T_{yy} = T_{zz} = \Big(\frac{1}{2} \dot{\phi}^2 - V(\phi) \Big) a^2. \tag{3.22}$$

Recall the perfect fluid energy-momentum tensor in cosmology

$$T_{ab} = \mathrm{diag}(\rho, a^2 P, a^2 P, a^2 P) \tag{3.23}$$

Therefore we can adopt the following notation for the energy density and the pressure of a homogeneous scalar field

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \tag{3.24}$$

$$P_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \tag{3.25}$$

The scalar field during inflation is often called the inflaton. $V(\phi)$ is the potential of the scalar field. Different inflationary models are described by different potentials $V(\phi)$.

By virtue of $P_\phi = w_\phi \rho_\phi$ we can define an effective equation of state for the scalar field. Note that w_ϕ is not a constant but also time-dependent

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.26}$$

The cosmological Einstein field equations are

$$H^2 = \frac{8\pi}{3}\rho, \qquad H = \frac{\dot{a}}{a} \tag{3.27}$$

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3P) \tag{3.28}$$

where we assume $k = 0$ and we neglected the cosmological term. We showed earlier that these equations imply the energy-momentum conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{3.29}$$

Let us now analyse the conservation equation for scalar fields

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0 \tag{3.30}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{3.31}$$

which is also called the scalar field's wave equation. It is common in inflation to rename the coupling constant

$$M_{\rm pl} = \frac{1}{\sqrt{8\pi}} \tag{3.32}$$

which is called the reduced Planck mass.

Using the reduced Planck mass we have

$$H^2 = \frac{1}{3}\frac{1}{M_{\rm pl}}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \tag{3.33}$$

$$-\frac{\ddot{a}}{a} = \frac{1}{3}\frac{1}{M_{\rm pl}}(\dot{\phi} - V) \tag{3.34}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{3.35}$$

One verifies that the condition for inflation is satisfied if

$$\dot{\phi}^2 < V(\phi) \tag{3.36}$$

If the potential is sufficiently flat, then the inflation condition $\dot{\phi}^2 < V(\phi)$ is satisfied. Even if it is not obeyed initially, this quickly changes if the field is away from the minimum of the potential.

The condition $\dot{\phi}^2 < V(\phi)$ physically speaking means that the kinetic energy should be small, slow motions.

3.5 Slow-roll inflation

The slow-roll approximation is the standard approximation to analyse inflation models. Physically slow motions require slow accelerations $\ddot{\phi} \ll 1$. Moreover the potential energy should be much larger than the kinetic energy $\frac{1}{2} \ll V(\phi)$.

In the approximation, the equations of the motion are

$$H^2 \simeq \frac{V(\phi)}{3M_{\rm pl}^2} \tag{3.37}$$

$$3H\dot{\phi} \simeq -V'(\phi) \tag{3.38}$$

The prime denotes differentiation with respect to the scalar field and $\simeq$ means equal within the approximation.

Two convenient parameters are the so-called slow-roll parameters

$$\epsilon(\phi) = \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta(\phi) = M_{\rm pl}^2 \frac{V''}{V} \tag{3.39}$$

The slow-roll approximation is valid if $\epsilon \ll 1$ and $\eta \ll 1$. Note that these conditions only restrict the form of the potential.

Example 3.1. Consider a massive inflaton field with potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{3.40}$$

we have

$$\frac{V'}{V} = \frac{m^2\phi}{\frac{1}{2}m^2\phi^2} = \frac{2}{\phi} \tag{3.41}$$

$$\frac{V''}{V} = \frac{m^2}{\frac{1}{2}m^2\phi^2} = \frac{2}{\phi^2} \tag{3.42}$$

Hence

$$\epsilon(\phi) = \frac{M_{\rm pl}^2}{2}\frac{4}{\phi^2} = 2\frac{M_{\rm pl}^2}{\phi^2} \ll 1 \tag{3.43}$$

$$\eta(\phi) = M_{\rm pl}^2 \frac{2}{\phi^2} = 2\frac{M_{\rm pl}^2}{\phi^2} \ll 1 \tag{3.44}$$

Both conditions are satisfied provided $\phi > \sqrt{2}M_{\rm pl}^2$. In general we may assume that inflation ends when $\eta = 1$.

Let us analyse if the slow-roll approximation always yields inflation. The Hubble parameter is defined by

$$H = \frac{\dot{a}}{a}, \qquad \dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2 \tag{3.45}$$

The inflation condition is

$$\ddot{a} > 0 \quad \Rightarrow \quad \dot{H} + H^2 > 0 \tag{3.46}$$

which is satisfied for positive $\dot{H}$ and we can write

$$-\frac{\dot{H}}{H^2} < 1. \tag{3.47}$$

On the other hand, using the slow-roll equations of motion to find

$$-\frac{\dot{H}}{H^2} \simeq \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \epsilon \tag{3.48}$$

Now, if the slow-roll approximation holds then $\epsilon \ll 1$ and therefore the inflation condition is satisfied.

3.6 e-foldings

The ratio of the scale factor at the end of the inflation to its value at some time t is usually used to measure the amount of inflation occurred. During inflation the scale factor typically increases by a factor of 10^{26}. Since this is a large number, one quantifies the amount of inflation by the number of e-foldings $\mathcal{N}$, defined by

$$\mathcal{N}(t) = \log\frac{a(t_{\rm end})}{a(t)} \tag{3.49}$$

Typically, the number of e-foldings is taken to be $50 - -60$. Note that at $t_{\rm end}$, we have $\mathcal{N}(t_{\rm end}) = 0$. Therefore, the number of e-foldings measures the amount of inflation that still has to occur after time t. For most inflation models, the number of e-foldings can be calculated from a given field value ϕ and given potential V (rather than from a given time or scale factor).

Using the definition of the Hubble parameter and the slow-roll approximation we find

$$\mathcal{N} = \log \frac{a(t_{\text{end}})}{a(t)} = \int_t^{t_{\text{end}}} H(t')dt' \simeq \frac{1}{M_{\text{pl}}^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'} d\phi \tag{3.50}$$

with $\epsilon(\phi_{\text{end}}) = 1$ to define the end of inflation.

Example 3.2. $V = \frac{m^2}{2}\phi^2$, $V' = m^2\phi$

$$\begin{aligned}\mathcal{N} &\simeq \frac{1}{M_{\text{pl}}^2} \int_{\phi_{\text{end}}}^{\phi_i} \frac{\frac{m^2}{2}\phi^2}{m^2\phi} d\phi = \frac{1}{M_{\text{pl}}^2} \frac{1}{4} \phi^2 \Big|_{\phi_{\text{end}}}^{\phi_i} \\ &\simeq \frac{1}{M_{\text{pl}}^2} \frac{1}{4} \left[\phi_i^2 - \phi_{end}^2\right] = \left(\frac{\phi_i}{2M_{\text{pl}}}\right)^2 - \frac{1}{2} \end{aligned} \tag{3.51}$$

For $\phi_i = 15M_{\text{pl}}$ one would find a little over 60 e-foldings. Note that we did not solve any field equations.

3.7 Power-law inflation

Power-law inflation is an exact solution which arises for the following potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{\text{pl}}}\right) \tag{3.52}$$

where V_0 and p are two constants.

Let us solve the slow-roll approximated field equations

$$H^2 \simeq \frac{V_0}{3M_{\text{pl}}^2} \exp\left(-\sqrt{\frac{2}{p}} \frac{p}{M_{\text{pl}}}\right) \tag{3.53}$$

$$3H\dot{\phi} \simeq +V_0\sqrt{\frac{2}{p}} \frac{1}{M_{\text{pl}}} \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{\text{pl}}}\right) \tag{3.54}$$

Eliminating H from the second equation yields

$$3\sqrt{\frac{V_0}{3}} \frac{1}{M_{\text{pl}}} \exp\left(-\frac{1}{\sqrt{2p}} \frac{\phi}{M_{\text{pl}}}\right) \dot{\phi} \simeq V_0\sqrt{\frac{2}{p}} \frac{1}{M_{\text{pl}}} \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{\text{pl}}}\right) \tag{3.55}$$

Separation of variable and integration leads to

$$\dot{\phi} \simeq \sqrt{2p} \log\left(\sqrt{\frac{V_0}{p(3p-1)}} \frac{t}{M_{\text{pl}}}\right) \tag{3.56}$$

Inserting $\frac{\phi}{M_{pl}}$ into the first equation yields

$$H^2 = \frac{p^2}{t^2} \quad \Rightarrow \quad a(t) = a_0 t^p. \tag{3.57}$$

Let us also compute the slow-roll parameters of this potential

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{\rm pl}}\right) \tag{3.58}$$

$$V'(\phi) = -\sqrt{\frac{2}{p}}\frac{1}{M_{\rm pl}} V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{\rm pl}}\right) \tag{3.59}$$

$$V''(\phi) = \frac{2}{p}\frac{1}{M_{\rm pl}} V_0 \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{\rm pl}}\right) \tag{3.60}$$

and hence

$$\epsilon = \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 = \frac{M_{\rm pl}^2}{2}\left(-\sqrt{\frac{2}{p}}\frac{1}{M_{\rm pl}}\right)^2 = \frac{1}{p} \tag{3.61}$$

$$\eta = M_{\rm pl}^2 \frac{V''}{V} = \frac{2}{p} \tag{3.62}$$

If $p > 1$ this solution satisfies the inflation conditions. However, since ϵ and η are both constants there is no natural end in this inflation model.

4 Introduction to cosmological perturbation theory[1]

Cosmological perturbation theory is an important tool for modern cosmology. It allows us, among other applications, to calculate the

- the growth of structure in the universe (structure formation)
- the fluctuations of the cosmic microwave background radiation

in an expanding background universe.

4.1 Local isometries and the Lie derivative

Let $p \in M$ be a point in the manifold M. Let us consider coordinates X^a and a new coordinate system X'^a such that $X'^a = X'^a(X^b)$. If g_{ab} denotes the metric on M, then there are two possible ways to interpret $g_{a'b'}(X'^c)$. In the active picture $p(X'^a)$ corresponds to another point $p' \in M$ while in the passive picture X'^a are the new coordinates of the same point p.

Definition 4.1. Local isometry. The transformation $X^a \mapsto X'^a = X'^a(X^b)$ is called a symmetry transformation or a local isometry if $g_{a'b'} = g_{ab}$.

Definition 4.2. Lie derivative of a function. Let $f \in C^\infty(M)$ and let ξ^a be a contravariant vector. Then the Lie derivative of f is defined by

$$L_\xi f = \xi^a \partial_a f. \tag{4.1}$$

Definition 4.3. Lie derivative of a contravariant vector. Let V^a and ξ^b be contravariant vectors. Then the Lie derivative of V^a with respect to ξ^b is defined by

$$(L_\xi V)^a = \xi^b \partial_b V^a - V^b \partial_b \xi^a. \tag{4.2}$$

This definition can be extended to higher rank tensors, in particular, the Lie derivative of the metric tensor g_{ab} is given by

$$(L_\xi g)_{ab} = \xi^c \partial_c g_{ab} + g_{cb} \partial_a \xi^c + g_{ac} \partial_b \xi^c \tag{4.3}$$

Theorem 4.1. *The vector ξ^a generates a local isometry if $(L_\xi g)_{ab} = 0$.*

[1]Please note that this section was completed only recently and hence may contain more typos than the first three sections.

Proof. Let ξ^a generate an infinitesimal transformation

$$X'^a = X^a + \xi^a. \tag{4.4}$$

Firstly, let us consider $g_{a'b'}(X'^c)$ and perform a Taylor expansion with respect to ξ^c, this yields

$$g_{a'b'}(X'^c) = g_{a'b'}(X^c) + \xi^c \partial_c g_{a'b'}(X^c) + O(\xi^c)^2. \tag{4.5}$$

Secondly, the metric g_{ab} being a rank 2 tensor, transforms under the coordinate transformation (4.4) such that

$$g_{a'b'}(X'^i) = \frac{\partial X^c}{\partial X'^a} \frac{\partial X^d}{\partial X'^b} g_{cd}(X^i). \tag{4.6}$$

From the coordinate transformation we find

$$\frac{\partial X^c}{\partial X'^a} = \frac{\partial (X'^c - \xi^c)}{\partial X'^a} = \delta^c_a - \partial_a \xi^c, \tag{4.7}$$

and therefore

$$\begin{aligned} \frac{\partial X^c}{\partial X'^a} \frac{\partial X^d}{\partial X'^b} &= (\delta^c_a - \partial_a \xi^c)(\delta^d_b - \partial_b \xi^d) \\ &= \delta^c_a \delta^d_b - \partial_a \xi^c \delta^d_b - \partial_b \xi^d \delta^c_a + O(\xi^c)^2. \end{aligned} \tag{4.8}$$

Taking these together we find

$$\begin{aligned} g_{a'b'}(X'^i) &= (\delta^c_a \delta^d_b - \partial_a \xi^c \delta^d_b - \partial_b \xi^d \delta^c_a) g_{cd}(X^i) + O(\xi^c)^2, \\ &= g_{ab}(X^i) - (\partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb}) + O(\xi^c)^2. \end{aligned} \tag{4.9}$$

By the definition of the local isometry we have $g_{a'b'} = g_{ab}$ and hence

$$\begin{aligned} g_{ab}(X^c) - g_{a'b'}(X^c) &= \xi^c \partial_c g_{a'b'} + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} \\ &= \xi^c \partial_c g_{ab} + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} = 0, \end{aligned} \tag{4.10}$$

where we used $\xi^c \partial_c g_{a'b'} = \xi^c \partial_c g_{ab} + O(\xi^c)^2$. Finally, comparison with the Lie derivative of the metric, we obtain

$$(L_\xi g)_{ab} = 0. \tag{4.11}$$

□

If we denote the transformed metric $g_{a'b'}(X^c)$ by $\tilde{g}$, then we can write

$$g - \tilde{g} = L_\xi g. \tag{4.12}$$

Example 4.1. Let us consider the vector $\xi^i = \delta_0^t$ and the Schwarzschild metric

$$g_{ab} = \mathrm{diag}(-(1-2M/r), (1-2M/r), r^2, r^2 \sin^2\theta), \tag{4.13}$$

with coordinates $X^a = (t, r, \theta, \phi)$. Inspection of the equations (4.3) immediately yields that all the three terms identically vanish and therefore $(L_\xi g)_{ab} = 0$ and hence $\xi^i = \delta_0^t$ generates a symmetry. $\xi^i = \delta_0^t$ corresponds to time translations and since the Schwarzschild spacetime is static we expect the presence of such a symmetry.

4.2 Killing vectors

From the previous subsection we know that a necessary condition for the existence of a local isometry is

$$\xi^c \partial_c g_{ab} + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} = 0. \tag{4.14}$$

Let us use the the condition $\nabla_c g_{ab} = 0$ to rewrite the partial derivative of the metric tensor. We have

$$0 = \nabla_c g_{ab} = \partial_c g_{ab} + \Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{ad}, \tag{4.15}$$

which gives

$$-\xi^c(\Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{ad}) + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} = 0. \tag{4.16}$$

This latter equation can now be written in terms of covariant derivatives and we get

$$\nabla_b \xi_a + \nabla_a \xi_b = 0. \tag{4.17}$$

These equations are called the Killing equations.

Since the second covariant derivatives of vectors involve the Riemann curvature tensor, the Killing vectors have to satisfy certain integrability conditions (We will not discuss these issues). Killing vectors are important in general relativity and cosmology because they can be used to define conserved quantities.

Example 4.2. Let us consider Minkowski spacetime, $g_{ab} = \text{diag}(-1, 1, 1, 1)$. The Killing equation becomes

$$\eta_{ca}\partial_b\xi^c + \eta_{cb}\partial_a\xi^c = 0, \tag{4.18}$$

and is solved by

$$\xi^c = \omega^c{}_b X^b + a^c, \tag{4.19}$$

where $\omega^c{}_b$ is skew-symmetric. Hence, for Minkowski spacetime there are 10 Killing vectors, this corresponds to the Poincaré group.

Example 4.3. The vector $\xi^a = \delta^a_t$ is a time-like Killing vector of the Schwarzschild spacetime.

4.3 Background metric and perturbations

Our working assumption is that spacetime (on large scales) is almost homogenous and isotropic and deviation are regarded small. The metric is consequently split into two parts, the background, and the perturbation.

As for the background line element, we consider the FLRW metric

$$\begin{aligned} ds^2 = {}^{(0)}g_{\alpha\beta}dX^\alpha dX^\beta &= dt^2 - a^2(t)\gamma_{ij}dx^i dx^j \\ &= a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j). \end{aligned} \tag{4.20}$$

Recall that η is conformal time, and that the spatial part of the metric is given by

$$\gamma_{ij} = \delta_{ij}\left(1 + k/4(x^2 + y^2 + z^2)\right)^{-2} \tag{4.21}$$

The background field equations in conformal time are

$$a'^2 + ka^2 = \frac{8\pi}{3}T^\eta_\eta a^4 \tag{4.22}$$

$$a'' + ka = \frac{4\pi}{3}Ta^3, \tag{4.23}$$

where the prime denotes the derivative with respect to conformal time and T denotes the trace of the energy-momentum tensor $T = T^\alpha_\alpha$. Note that the field equations imply the energy-momentum conservation equation, which for this choice of coordinates becomes

$$\frac{dT^\eta_\eta}{d\ln a} + (4T^\eta_\eta - T) = 0. \tag{4.24}$$

Next, we include the perturbations. Let us write the complete line element as

$$ds^2 = {}^{(0)}g_{\alpha\beta}dx^\alpha dx^\beta + \delta g_{\alpha\beta}dx^\alpha dx^\beta, \tag{4.25}$$

where $\delta g_{\alpha\beta}$ denotes the perturbation. The complete metric can be written such that $g_{\alpha\beta} = {}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}$.

The perturbations $\delta g_{\alpha\beta}$ can be classified according to their transformation behaviour under purely spatial coordinate transformation on constant time hypersurfaces. There are three types of perturbations: scalar, vector and tensor perturbations. In general the vector perturbations decay in an expanding universe while the tensor perturbations yield gravitational waves. These waves do not couple to the energy density and pressure inhomogeneities. Only the scalar perturbations may yield growing inhomogeneities which are important in the context of structure formation. Let us write

$$\delta g_{\alpha\beta} = \begin{pmatrix} \delta g_{\eta\eta} & \delta g_{\eta i} \\ \delta g_{i\eta} & \delta g_{ij} \end{pmatrix} \tag{4.26}$$

The term $\delta g_{\eta\eta}$ is a scalar quantity, while we can think of the term $\delta g_{\eta i}$ as being a vector. However, we can write $\delta g_{\eta i} = \nabla_i f + V_i$, this means we can decompose $\delta g_{\eta i}$ into a scalar and a vector part. Similarly, the spatial part of the perturbed metric can be decomposed into a scalar, vector and tensor part $\delta g_{ij} = \nabla_{ij} f + \nabla_i W_j + U_{ij}$. Here $\nabla_{ij} f = \nabla_i \nabla_j f$.

4.3.1 Scalar perturbations

The most general form of the scalar perturbations is given by

$$\delta g^{(\mathrm{s})}_{\alpha\beta} = a^2(\eta) \begin{pmatrix} 2\phi & -\nabla_i B \\ -\nabla_i B & 2(\psi\gamma_{ij} - \nabla_{ij} E) \end{pmatrix}, \tag{4.27}$$

where ϕ, ψ, E and B are functions of space and time. Therefore, the complete line element with scalar perturbations reads

$$\begin{aligned} ds^2 = a^2(\eta)\big[(1 + 2\phi)d\eta^2 - 2\nabla_i B dx^i d\eta \\ - ((1 - 2\psi)\gamma_{ij} + 2\nabla_{ij} E)dx^i dx^j\big]. \end{aligned} \tag{4.28}$$

4.3.2 Vector perturbations

The most general form of vector perturbations is given by

$$\delta g^{(\mathrm{s})}_{\alpha\beta} = -a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & \nabla_i F_j + \nabla_j F_i \end{pmatrix}, \tag{4.29}$$

where F_i, and S_i two three-vector which are divergence-less

$$\nabla^i F_i = 0, \qquad \nabla^i S_i = 0. \tag{4.30}$$

4.3.3 Tensor perturbations

The most general form of tensor perturbations is given by

$$\delta g^{(\mathrm{s})}_{\alpha\beta} = -a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \tag{4.31}$$

where h_{ij} is a symmetric three tensor which satisfies

$$h^i{}_i = 0, \qquad \nabla^j h_{ij} = 0. \tag{4.32}$$

Remark. The scalar perturbations are characterised by four scalar functions (4 functions), the vector perturbations are described by two traceless three-vector (4 functions, 6 functions minus 2 constraints), and finally the tensor perturbations contain one symmetric three-tensor (2 functions, 6 functions minus 4 constraints). Therefore, the total number is ten, and this number agrees with the number of independent function for an arbitrary metric.

4.4 Gauge invariant variables

Let us consider the following infinitesimal coordinate transformation which preserves the scalar nature of the metric perturbations

$$\eta \mapsto \tilde{\eta} = \eta + \xi(\eta, x^i), \tag{4.33}$$

$$x^i \mapsto \tilde{x}^i = x^i + \gamma^{ij} \nabla_j \xi(\eta, x^i). \tag{4.34}$$

From Section 4.1 we now that the metric's change under infinitesimal coordinate transformations is described by the Lie derivative. Keeping terms in lowest order only, and considering linear perturbations, we have

$$\delta g_{ab} - \delta \tilde{g}_{ab} = (L_\xi g)_{ab}. \tag{4.35}$$

The new functions $\tilde{\phi}$ etc. are given by

$$\tilde{\phi} = \phi - \frac{a'}{a}\xi^0 - \xi^{0\prime}, \qquad \tilde{\psi} = \psi + \frac{a'}{a}\xi^0, \tag{4.36}$$

$$\tilde{B} = B + \xi^0 - \xi', \qquad \tilde{E} = E - \xi. \tag{4.37}$$

By suitably combining the four functions ϕ, ψ, B and E, we can construct gauge invariant variable which do not change under the above infinitesimal coordinate transformation. The simplest and most used variables are

$$\Phi = \phi + \frac{1}{a}[(B - E')a]', \qquad \Psi = \psi - \frac{a'}{a}(B - E'). \tag{4.38}$$

In conformal Newtonian gauge, or longitudinal gauge, one chooses $B = E = 0$, such that the metric perturbations are uniquely described by the two functions Φ and Ψ. In that gauge the metric is

$$ds^2 = a^2(\eta)\left((1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\gamma_{ij}dx^i dx^j\right). \tag{4.39}$$

Note that even a scalar function $q(\eta, x^i)$ is not a gauge invariant quantity. Let us consider $q(\eta, x^i) = q_0(\eta) + \delta q(\eta, x^i)$. Then, we find

$$\delta\tilde{q} = \delta q - L_\xi q_0 = \delta q - \xi^a \partial_a q_0 = \delta q - \xi^0 q_0'. \tag{4.40}$$

Using the equations for $\tilde{B}$ and $\tilde{E}$ we can construct a gauge invariant scalar by noting

$$\tilde{B} - \tilde{E}' = B - E' + \xi^0, \tag{4.41}$$

and therefore

$$\delta q^{(\mathrm{gi})} = \delta q + (B - E')q_0'. \tag{4.42}$$

To see this

$$\begin{aligned}\widetilde{\delta q}^{(\mathrm{gi})} &= \widetilde{\delta q} + (\tilde{B} - \tilde{E}')q_0' = \delta q - \xi^0 q_0' + (B - E' + \xi^0)q_0' \\ &= \delta q + (B - E')q_0' = \delta q^{(\mathrm{gi})}\end{aligned} \tag{4.43}$$

4.5 Perturbed field equations

4.5.1 Perturbed Einstein tensor components

Starting from (4.39) with $k = 0$ for simplicity, we can now compute all non-vanishing Christoffel symbol components and subsequently all Ricci tensor components and finally the Einstein tensor components. Recall that we are only interested in linear perturbations, this means that we can neglect all higher order terms.

Example 4.4. Let us compute the component $\Gamma^\eta_{\eta\eta}$ of the Christoffel symbol. By definition

$$\begin{aligned}\Gamma^\eta_{\eta\eta} = \frac{1}{2} g^{\eta\eta}\partial_\eta g_{\eta\eta} &= \frac{1}{2}\left(a^2(1+2\Phi)\right)^{-1}\partial_\eta\left(a^2(1+2\Phi)\right)\\ &= \frac{1}{2}a^{-2}(1+2\Phi)^{-1}(2aa'(1+2\Phi)+a^2 2\Phi')\\ &= (1+2\Phi)^{-1}\left(\frac{a'}{a}(1+2\Phi)+\Phi'\right)\\ &\simeq (1-2\Phi)\left(\frac{a'}{a}(1+2\Phi)+\Phi'\right) \simeq \frac{a'}{a}+\Phi', \end{aligned} \tag{4.44}$$

plus higher order terms.

For the background components of the Einstein tensor we find

$$^{(0)}G^0_0 = 3a'^2/a^4, \tag{4.45}$$

$$^{(0)}G^i_0 = 0, \tag{4.46}$$

$$^{(0)}G^j_i = (2a''a - a'^2)/a^4\,\delta^j_i, \tag{4.47}$$

while the first order perturbed components are given by

$$\delta G^0_0 = 2\Big[-3\frac{a'}{a}(\Psi' + a'/a\,\Phi) + \Delta\Psi\Big]/a^2, \tag{4.48}$$

$$\delta G^0_i = 2\partial_i\Big[\frac{a'}{a}\Phi + \Psi'\Big]/a^2. \tag{4.49}$$

The final term δG^j_i reads

$$\begin{aligned}\delta G^j_i = -2\Big[&\Big\{\Psi'' + \frac{a'}{a}(\Phi' + 2\Psi') + 2\frac{a''}{a}\Phi - \frac{a'^2}{a^2}\Phi\Big\}\delta^j_i\\ &+\Big\{\frac{1}{2}\Delta(\Phi-\Psi)\Big\}\delta^j_i - \frac{1}{2}\partial_i\partial^j(\Phi-\Psi)\Big]/a^2. \end{aligned}\tag{4.50}$$

Note that the off-diagonal components of the perturbed Einstein tensor do not vanish, indeed

$$\delta G^j_i = \partial_i\partial^j(\Phi-\Psi)/a^2, \qquad i \neq j. \tag{4.51}$$

4.5.2 Gravity waves

Let us consider tensor perturbations in the form

$$g_{\alpha\beta} = a^2(\eta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+h_+) & -h_\times & 0 \\ 0 & -h_\times & -(1-h_-) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.52}$$

The perturbed Einstein field equations yield two independent equations

$$\delta G^x_x = -\left[h''_+ + 2\frac{a'}{a}h'_+ - \partial_{zz}h_+\right]/(2a^2) = 0 \tag{4.53}$$

When analysing wave equations and perturbations it is often convenient to work in 'Fourier space.' Let us define the Fourier transform by

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{\vec{k}\cdot\vec{x}} f(\vec{k}), \tag{4.54}$$

where $\vec{x} = \{x, y, z\}$ and $\vec{k}\cdot\vec{x} = xk_x + yk_y + zk_z$. Note that we use the same symbol for f and its Fourier transform. The probably most important feature of the Fourier transform is

$$\partial_{x^i} f(\vec{x}) = k_i \int \frac{d^3k}{(2\pi)^3} e^{\vec{k}\cdot\vec{x}} f(\vec{k}) \tag{4.55}$$

Now, the above wave equation can be written as

$$h''_+ + 2\frac{a'}{a}h'_+ + k_z^2 h_+ = 0. \tag{4.56}$$

This corresponds to gravitational waves travelling in the $\pm z$ direction. If the universe is static ($\dot{a} = 0$) the wave's amplitude does not change over time. However, in an expanding universe the oscillations damp out. The amplitude of gravitational waves present in the early universe is extremely small today.

4.5.3 Matter perturbations

Let us assume that the background dynamics is determined by a perfect fluid characterised by an energy density ρ, pressure P and 4-velocity u^α. Its energy-momentum tensor reads

$$T^\beta_\alpha = (\rho + P)u_\alpha u^\beta - P\delta^\beta_\alpha. \tag{4.57}$$

By considering scalar perturbations, the first order fluid perturbations have the form

$$\delta T_\alpha^\beta = \begin{pmatrix} \delta\rho & -({}^{(0)}\rho + {}^{(0)}P)\partial_i V/a \\ ({}^{(0)}\rho + {}^{(0)}P)\partial_i V\, a & -\delta P \delta_{ij} + \nabla_{ij}\sigma \end{pmatrix}. \tag{4.58}$$

$\delta\rho$ and δP are the perturbed energy density and pressure respectively. V is the potential of the 3-velocity and σ is the anisotropic stress. In general the pressure P is not only a function of the energy density but also depends on the the entropy (we will neglect this henceforth and consider only adiabatic perturbations).

In the absence of anisotropic stresses $\sigma = 0$, it follows immediately that the spatial part of the perturbed energy-momentum tensor is diagonal. By virtue of the field equations

$$\delta G_i^j = 8\pi \delta T_i^j = 0, \qquad i \neq j, \tag{4.59}$$

this leads to

$$\partial_i \partial^j (\Phi - \Psi)/a^2 = 0 \quad \Rightarrow \quad \Phi = \Psi, \tag{4.60}$$

which considerably simplifies the perturbed field equations. The perturbed metric in that case is characterised by only one function.

4.5.4 Scalar field perturbation

Assuming the background scalar field to be homogeneous and isotropic, we write

$$\varphi(\eta, x, y, z) = \varphi_0(\eta) + \delta\varphi(\eta, x, y, z), \tag{4.61}$$

which yields the following background energy-momentum tensor

$${}^{(0)}T_\eta^\eta = \frac{1}{2a^2}{\varphi_0'}^2 + V(\varphi_0) = \rho_\varphi, \tag{4.62}$$

$${}^{(0)}T_i^0 = 0, \tag{4.63}$$

$${}^{(0)}T_i^j = \frac{1}{2a^2}{\varphi_0'}^2 - V(\varphi_0) = P_\varphi. \tag{4.64}$$

For the first order perturbations we obtain

$$\delta T_\eta^\eta = \left[-{\varphi_0'}^2\Phi + \varphi_0'\delta\varphi' + V_\varphi a^2 \delta\varphi\right]/a^2, \tag{4.65}$$

$$\delta T_i^0 = \varphi_0' \partial_i \delta\varphi / a^2, \tag{4.66}$$

$$\delta T_i^j = \left[{\varphi_0'}^2\Phi - \varphi_0'\delta\varphi' + V_\varphi a^2 \delta\varphi\right]\delta_i^j. \tag{4.67}$$

Having worked out all quantities in first order perturbation theory, we can also compute the perturbed energy-momentum conservation equation $\nabla_\alpha T^\alpha_\beta = 0$. This equation (only $\beta = \eta$ yields something non-trivial) is given by

$$\delta\varphi'' + 2aH\delta\varphi' + k^2\delta\varphi + V_{\varphi\varphi}a^2\delta\varphi = 0, \tag{4.68}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$.

Typically for inflation models the term $V_{\varphi\varphi}$ is of the order of the slow-roll parameters and hence is negligible in the slow-roll approximation. Hence, the resulting wave equation takes the form

$$\delta\varphi'' + 2aH\delta\varphi' + k^2\delta\varphi = 0. \tag{4.69}$$

Like in the case for gravity waves, this equation corresponds to a damped harmonic oscillator.

4.5.5 Example: Perturbations in the matter dominated universe

Let us consider perturbations in the matter dominated universe. The background is defined by $a(\eta) = a_0\eta^2$, ${}^{(0)}\rho(\eta) = \rho_0/a^3$ and ${}^{(0)}P = 0$. We assume isotropic, adiabatic perturbations which are also dust-like, $\delta P = 0$. The absence of anisotropic stresses yields $\Phi = \Psi$. Since $\delta P = 0$, the spatial component of the perturbed field equations is given by

$$\Psi'' + 3\frac{a'}{a}\Psi' + 2\frac{a''}{a}\Psi - \frac{a'^2}{a^2}\Psi = 0. \tag{4.70}$$

Since $2a''/a = a'^2/a^2$ and $a'/a = 2/\eta$ in the matter dominated universe, the perturbation Ψ satisfies the equation

$$\Psi'' + \frac{6}{\eta}\Psi' = 0, \tag{4.71}$$

which is solved by

$$\Psi(\eta, x, y, z) = \Psi_0(x, y, z) + \frac{\Psi_1(x, y, z)}{\eta^5}. \tag{4.72}$$

Since we are interested in the non-decaying mode only, we set $\Psi_1 = 0$.

Next, we consider the (00)-component of the perturbed field equations. Using that $\Delta\Psi_0 = -k^2\Psi_0$ in Fourier space, we find

$$-\frac{2}{a_0^2}\frac{12 + k^2\eta^2}{\eta^6}\Psi_0 = 8\pi\delta\rho. \tag{4.73}$$

To interpret this equation it is convenient to divide by the background energy density, which defines the density contrast

$$\delta = \frac{\delta\rho}{{}^{(0)}\rho} = -\frac{a_0}{4\pi\rho_0}(12 + k^2\eta^2)\Psi_0. \tag{4.74}$$

Recall that the scale factor is $a(\eta) = a_0\eta^2$ and hence the perturbations grow with the expansion of the universe. For a 'proper' instability we would have expected an exponential growth rate.

4.6 The power spectra of perturbations

4.6.1 The harmonic oscillator

The harmonic oscillator is described by the following differential equation

$$\ddot{x} + \omega^2 x = 0. \tag{4.75}$$

When quantised, the position x becomes the position operator $\hat{x}$

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger, \tag{4.76}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operator, respectively; $v = \exp(-i\omega t)/\sqrt{2\omega}$. They satisfy the following commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, while the other commutation relations between the a's vanish.

To compute the power spectrum we need to know that the variance of $\hat{x}$ is

$$\langle |\hat{x}|^2 \rangle = |v(\omega, t)|^2, \tag{4.77}$$

which for the harmonic oscillator yields $\langle |\hat{x}|^2 \rangle = 1/2\omega$.

A quantum field can be regarded as a system of infinitely many harmonic oscillators, each one being at a different position in space. Going to Fourier space, each of these oscillators has a different wave vector $\vec{k}$. Let use write

$$\hat{X}(\vec{k}, \eta) = v(k, \eta)\hat{a}_{\vec{k}} + v^*(k, \eta)\hat{a}^\dagger_{\vec{k}} \tag{4.78}$$

The variance of perturbations in X defines the power spectrum

$$\langle \hat{X}^\dagger(\vec{k}, \eta)\hat{X}(\vec{k}', \eta) \rangle = |v(\vec{k}, \eta)|^2 (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \tag{4.79}$$

$$= (2\pi)^3 P_X(k) \delta^3(\vec{k} - \vec{k}'), \tag{4.80}$$

where $P_X(k)$ is the power spectrum.

4.6.2 The scalar field case

Consider the perturbed scalar field equation

$$\delta\varphi'' + 2aH\delta\varphi' + k^2\delta\varphi = 0. \tag{4.81}$$

Let us introduce a new function $v = a\delta\varphi$. This leads to

$$\delta\varphi = \frac{v}{a}, \qquad \delta\varphi' = \frac{v'}{a} - \frac{va'}{a^2}, \tag{4.82}$$

$$\delta\varphi'' = \frac{v''}{a} - 2\frac{va'}{a^2} - \frac{va''}{a^2} + 2\frac{va'^2}{a^3}. \tag{4.83}$$

Therefore, we find

$$\begin{aligned}\frac{v''}{a} - 2\frac{va'}{a^2} - \frac{va''}{a^2} + 2\frac{va'^2}{a^3} + 2\frac{a'}{a}\Big(\frac{v'}{a} - \frac{va'}{a^2}\Big) + k^2\frac{v}{a} \\ = \frac{1}{a}\Big[v'' + \big(k^2 - \frac{a''}{a}\big)v\Big] = 0.\end{aligned} \tag{4.84}$$

Hence, the scalar field's perturbations satisfies the equation

$$v'' + \big(k^2 - \frac{a''}{a}\big)v = 0. \tag{4.85}$$

We are interested in the power spectrum of the fluctuations $\delta\varphi$, so

$$\begin{aligned}\langle\widehat{\delta\varphi}^\dagger(\vec{k},\eta)\widehat{\delta\varphi}(\vec{k}',\eta)\rangle = \frac{1}{a^2}\langle\hat{v}^\dagger(\vec{k},\eta)\hat{v}(\vec{k}',\eta)\rangle \\ = \frac{|v(\vec{k},\eta)|^2}{a^2}(2\pi)^3\delta^3(\vec{k}-\vec{k}').\end{aligned} \tag{4.86}$$

Comparison with the definition of the power spectrum, we conclude

$$P_\varphi(k) = \frac{|v(\vec{k},\eta)|^2}{a^2}. \tag{4.87}$$

In order to find the power spectrum, we need to solve Eq. (4.85). To do so, let us first of all estimate the function a''/a using the slow-roll approximation of scalar field inflation. Recall that the slow-roll parameter ϵ can be defined using the Hubble parameter

$$\epsilon = \frac{d}{dt}\frac{1}{H} = \frac{H'}{aH^2}. \tag{4.88}$$

Conformal time is defined by

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H}. \tag{4.89}$$

If we assume $\epsilon \ll 1$, this is equivalent with assuming $1/H$ to be slowly varying and hence we can approximate conformal time by

$$\eta \simeq \frac{1}{H} \int \frac{da}{a^2} = -\frac{1}{H}\frac{1}{a}. \tag{4.90}$$

This means that we can write $a' = a^2 H \simeq -a/\eta$. Hence,

$$\frac{a''}{a} \simeq -\frac{1}{a}\frac{d}{d\eta}\Big(\frac{a}{\eta}\Big) = -\frac{1}{a}\frac{a'\eta - a}{\eta^2} \simeq -\frac{1}{a}\frac{(-2a)}{\eta^2} = \frac{2}{\eta^2}. \tag{4.91}$$

So we are able to estimate the behaviour of a''/a for all slow-roll inflation models, again without solving field equations.

The equation for the perturbations now takes the form

$$v'' + \big(k^2 - \frac{2}{\eta^2}\big)v = 0. \tag{4.92}$$

Such equations are in general solved by Bessel functions. We are interested in a properly normalised solution which becomes the simple harmonic oscillator when $-\eta$ is large ($e^{-ik\eta}/\sqrt{2k}$), this means before inflation has done most of its work and when the mode is well within the horizon. This solution is given by

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\Big(1 - \frac{i}{k\eta}\Big). \tag{4.93}$$

During inflation $-k\eta$ becomes smaller and hence we can find the power spectrum after inflation has work for many e-foldings by considering

$$\frac{e^{-ik\eta}}{\sqrt{2k}}\Big(1 - \frac{i}{k\eta}\Big) \quad \xrightarrow[-k\eta \ll 1]{} \quad \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{-i}{k\eta}. \tag{4.94}$$

Therefore the primordial spectrum of fluctuations are given by

$$P_\varphi(k) = \frac{1}{a^2}\frac{1}{2k^3\eta^2} = \frac{H^2}{2k^3}, \tag{4.95}$$

where we used the above estimate that $1/\eta^2 \simeq a^2 H^2$ in slow-roll approximation.